Natural Deduction for Linear Logic

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Keywords: Linear logic, multiplicative fragment, natural deduction, multiple-conclusion rules, proof theory.

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1 Introduction

Linear logic, proposed by Girard [Gir1987], has shown to be extremely valuable for the formalization of typical problems in Computer Science. This is due to the fact that, in linear logic, sentences are treated as resources or actions, and logical consequences as transitions among states. In [Gir1995], Girard presents the following example where the sentential signs A_1, A_2 and A_3 stand for actions:

A_1: to spend \$1.
A_2: to buy a packet of Camels.
A_3: to buy a packet of Malboro.

Suppose that the packet of Camels and the packet of Malboro cost \$1 each. In linear logic, the action $(A_1 \multimap A_2)$ means to spend \$1 to buy a packet of Camels. With \$1, we bought either a packet of Camels or a packet of Malboro and, after buying them, we will not have \$1 anymore. Thus, in linear logic, from (A_1) we can get $(A_1 \multimap A_2)$ or $(A_1 \multimap A_3)$, but not both.

Since linear logic has a great number of connectives, the deductive power of linear signs is hard to understand. For this reason, many syntactical and semantical approaches have been developed for linear logic as, for instance, the linear sequent calculus [Gir1987] although a natural deduction system for full linear logic has not been proposed yet.

The natural deduction system, introduced by Gentzen [Gen1934] for classical logic, is a syntactical method that carries quite a meaning in his rules. When Gentzen designed his rules, he tried to bring closer the natural deduction proofs to the human reasoning.

The very aim of this work is to develop a natural deduction system for linear logic bearing in mind the features of the method proposed by Gentzen [Gen1934]. Furthermore, many proof-theorectical results can be further investigated such as normalization theorems proposed by Prawitz [Pra1965] for classical logic within the context of natural deduction systems.

Among the corollaries of normalization theorems, it can be cited the definition of canonical proofs, the subformula principle and the consistency proof of the calculus. Moreover, normalization theorems for the natural deduction system in classical logic have stimulated the development of important studies about length of proofs. The results obtained from these studies are considered better than those obtained from similar studies in sequent calculus [Per1982]. Such results can be applied in automatic theorem provers and in programming languages based on logic.

Besides the sequent calculus, Girard [Gir1987,Gir1995] proposed a new deduction system named proof-nets. Proof-nets are considered as to be the natural deduction of linear sequent calculus. They are defined as unoriented connected graphs called proof-strutures. However, in order to recognize a proof-net, a soundness criterion must be applied to a proof-structure, which overload the task of building up a logically correct deduction within this system.

In [Med2001], a natural deduction system for a subset of the multiplicative fragment of linear logic was proposed. She also proved normalization theorems and used them to define translations between logics. The natural deduction system that will be presented here takes Medeiros's system as a starting point. However, some rules are modified and others are introduced to cope with all multiplicative fragment. In section 2, linear logic is presented through a sequent calculus. Our linear natural deduction system is introduced in section 3 and completeness and correctness is proved in section 4. Conclusions and further works are pointed out in the last section.

2 Linear Logic

2.1 The Language of Linear Logic

The language $\mathcal{L}$ of sentential linear logic involves connectives for two different conjunctions "times" ($\otimes$) and "with" ($\&$), for two different disjunctions "par" ($⅋$) and "plus" ($\oplus$), for linear implication ($\multimap$) and negation ($^{\perp}$), besides two exponential connectives "of course" ($!$) and "why not" ($?$). The alphabet $\mathcal{A}$ of this language includes, in addition to the above signs, sentential signs $(A_1, A_2, A_3, \ldots)$, auxiliary signs ((,)), constants for linear contradiction ($\perp$) and for linear tautology (1). We assume that none of the signs of $\mathcal{A}$ are a finite sequence of other signs.

We will use small Greek letters ($\alpha, \beta, \gamma, \ldots$) to represent well formed formulas (wffs or simply formulas), and capital Greek letters (Γ, Δ, Θ, ...), but Π and Σ, to represent sets of wffs. We inductively define the set of wffs as follows:

i) Every sentential sign is a wff.

ii) $\perp$ is a wff.
iii) 1 is a wff.
iv) If α and β are wffs, then $(\alpha^{\perp})$, $(!\alpha)$, $(?\alpha)$, $(\alpha \otimes \beta)$, $(\alpha \& \beta)$, $(\alpha \parr \beta)$, $(\alpha \oplus \beta)$, $(\alpha \multimap \beta)$ are also wffs.
v) None of the expressions generated from the signs of the alphabet $\mathcal{A}$, besides those obtained from i), ii), iii), and iv), can be considered wffs.

2.2 The Sequent Calculus for Linear Logic

In the sequent calculus for linear logic proposed by Girard [Gir1987,Gir1995], "Weakening" and "Contraction" structural rules are suppressed since, through them, it could be possible to discharge hypotheses in an unrestricted way. Linear logic treats hypotheses as limited and relevant resources: they must be discharged for once only.

However, in exceptional cases, resources can be reused and/or discharged without being used. Whenever it is the case, exponential operators will be used prefixing formulas that represent unlimited resources and "Weakening"/"Contraction" rules will be reintroduced in a restricted way.

Rules for linear sequent calculus are as follows:

Identity $\quad \dfrac{}{\alpha \vdash \alpha} \qquad \dfrac{}{\vdash 1}\quad$ Tautology

Contradiction $\quad \dfrac{}{\perp \vdash} \qquad \dfrac{\Gamma' \vdash \alpha, \Delta' \quad \Gamma'', \alpha \vdash \Delta''}{\Gamma', \Gamma'' \vdash \Delta', \Delta''}\quad$ Cut

$\mathrm{L}_1 \quad \dfrac{\Gamma \vdash \Delta}{\Gamma, 1 \vdash \Delta} \qquad \dfrac{\Gamma \vdash \Delta}{\Gamma \vdash \perp, \Delta}\quad \mathrm{R}_\perp$

L_x (Exchange) $\quad \dfrac{\Gamma, \beta, \alpha \vdash \Delta}{\Gamma, \alpha, \beta \vdash \Delta} \qquad \dfrac{\Gamma \vdash \beta, \alpha, \Delta}{\Gamma \vdash \alpha, \beta, \Delta}\quad$ R_x (Exchange)

$\mathrm{L}_\otimes \quad \dfrac{\Gamma, \alpha, \beta \vdash \Delta}{\Gamma, (\alpha \otimes \beta) \vdash \Delta} \qquad \dfrac{\Gamma' \vdash \alpha, \Delta' \quad \Gamma'' \vdash \beta, \Delta''}{\Gamma', \Gamma'' \vdash (\alpha \otimes \beta), \Delta', \Delta''}\quad \mathrm{R}_\otimes$

$\mathrm{L}_\parr \quad \dfrac{\Gamma', \alpha \vdash \Delta' \quad \Gamma'', \beta \vdash \Delta''}{\Gamma', \Gamma'', (\alpha \parr \beta) \vdash \Delta', \Delta''} \qquad \dfrac{\Gamma \vdash \alpha, \beta, \Delta}{\Gamma \vdash (\alpha \parr \beta), \Delta}\quad \mathrm{R}_\parr$

$\mathrm{L}_{1\&} \quad \dfrac{\Gamma, \alpha \vdash \Delta}{\Gamma, (\alpha \& \beta) \vdash \Delta} \qquad \dfrac{\Gamma \vdash \alpha, \Delta \quad \Gamma \vdash \beta, \Delta}{\Gamma \vdash (\alpha \& \beta), \Delta}\quad \mathrm{R}_\&$

$\mathrm{L}_{2\&} \quad \dfrac{\Gamma, \beta \vdash \Delta}{\Gamma, (\alpha \& \beta) \vdash \Delta} \qquad \dfrac{\Gamma \vdash \alpha, \Delta}{\Gamma \vdash (\alpha \oplus \beta), \Delta}\quad \mathrm{R}_{1\oplus}$

$\mathrm{L}_\oplus \quad \dfrac{\Gamma, \alpha \vdash \Delta \quad \Gamma, \beta \vdash \Delta}{\Gamma, (\alpha \oplus \beta) \vdash \Delta} \qquad \dfrac{\Gamma \vdash \beta, \Delta}{\Gamma \vdash (\alpha \oplus \beta), \Delta}\quad \mathrm{R}_{2\oplus}$

$$\mathrm{L}_{\multimap}\ \frac{\Gamma', \vdash \alpha, \Delta' \quad \Gamma'', \beta \vdash \Delta''}{\Gamma', \Gamma'', (\alpha \multimap \beta) \vdash \Delta', \Delta''} \qquad \frac{\Gamma, \alpha \vdash \beta, \Delta}{\Gamma \vdash (\alpha \multimap \beta), \Delta}\ \mathrm{R}_{\multimap}$$

$$\mathrm{L}_{\perp} \quad \frac{\Gamma \vdash \alpha, \Delta}{\Gamma, (\alpha^{\perp}) \vdash \Delta} \qquad \frac{\Gamma, \alpha \vdash \Delta}{\Gamma \vdash (\alpha^{\perp}), \Delta} \quad \mathrm{R}_{\perp}$$

$$\mathrm{L}_{c!}\ (\text{Contraction}) \quad \frac{\Gamma, (!\alpha), (!\alpha) \vdash \Delta}{\Gamma, (!\alpha) \vdash \Delta} \qquad \frac{\Gamma \vdash (?\alpha), (?\alpha), \Delta}{\Gamma \vdash (?\alpha), \Delta} \quad \mathrm{R}_{c?}\ (\text{Contraction})$$

$$\mathrm{L}_{w!}\ (\text{Weakening}) \quad \frac{\Gamma \vdash \Delta}{\Gamma, (!\alpha) \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash (?\alpha), \Delta} \quad \mathrm{R}_{w?}\ (\text{Weakening})$$

$$\mathrm{L}_{!}\ (\text{Dereliction}) \quad \frac{\Gamma, \alpha \vdash \Delta}{\Gamma, (!\alpha) \vdash \Delta} \qquad \frac{\Gamma \vdash \alpha, \Delta}{\Gamma \vdash (?\alpha), \Delta} \quad \mathrm{R}_{?}\ (\text{Dereliction})$$

$$\mathrm{L}_{?}\ (\text{Why not?}) \quad \frac{!\Gamma, \alpha \vdash ?\Delta}{!\Gamma, (?\alpha) \vdash ?\Delta} \qquad \frac{!\Gamma \vdash \alpha, ?\Delta}{!\Gamma \vdash (!\alpha), ?\Delta} \quad \mathrm{R}_{!}\ (\text{Of course!})$$

$!\Gamma$ means that all wff in the set Γ of formulas is prefixed by "!". $?\Gamma$ is equally defined, *mutatis mutandis*.

In this paper, sequents will be treated as a "multiset" [1] and not as a sequence of formulas. Hence, "L_x" and "R_x" rules will not be used.

Deductions in sequent calculus can be described as trees on which vertices are sequents and edges represent the application of rules to sequents. The sequents of "Identity", "Contradiction" and "Tautology" must appear at the top of the tree. We will use the capital Greek letters Π and Σ, eventually indexed, to represent deductions.

We take for granted the notion of an occurrence of a formula or (synonymously) a formula occurrence in a deduction. Two formula occurrences are said to be of the same form or shape if they are occurrences of the same formula; they are identical only if they also stand at the same place in the deduction.

2.3 The Multiplicative and Additive Fragments

In the sequent calculus, "$\mathrm{R}_{\&}$" and "$\mathrm{L}_{\oplus}$" rules share the same context Γ and Δ. Because of this feature, the connectives "&" and "$\oplus$" are called additives. Contrary to this, "$\otimes$" and "$⅋$" are named multiplicative connectives.

Linear logic can be divided into the multiplicative fragment and the additive fragment, each one of these containing one conjunction and one disjunction. The exponential operators and the linear negation belong to both fragments. The implication "$\multimap$" is considered multiplicative because it can be obtained through "$⅋$" and "$^{\perp}$". The constants "$\perp$" and "1" are also considered as multiplicative.

[1] "Multisets" of formulas are sets of formulas where two occurrences of the same formula are considered as distinct elements.

3 Natural Deduction for Linear Logic

According to Prawitz [Pra1965], a natural deduction system can be defined as a set of rules that determine the concept of deduction for a language. The rules for such a system should give the idea of “natural” as intended by Gentzen and are of two kind: introduction and elimination for each sign of the language. These rules indicate, in atomic steps, how to use a logical sign in a deduction.

The language for linear logic used here was defined in section 2.1 and our proposed rules will be given below. Correctness and completeness of our deductive system will be proved in relation to Girard’s sequent system.

3.1 Natural Deduction for the Multiplicative Fragment

In [Med2001], Medeiros has presented a natural deduction system, named NLLM, correct and complete with respect to Girard’s sequent system, for a subset of the multiplicative fragment of linear logic. She has introduced the multiplicative constants and connectives “1”, “$^\perp$”, “$\parr$”, “?” by definition. Starting on some of the Medeiros ideas, we have created a calculus for all multiplicative fragment called NDMF.

Using NDMF system, we can prove a wff α from a set of wffs Γ. In this case, we use the notation $\Gamma \vdash_{NDMF} \alpha$. The set Γ represents assumptions. In the NDMF system, the constant “1” is defined as “$(\perp \multimap \perp)$”.

The inference rules of NDMF calculus are defined as follows:

$$\mathrm{I}_{\multimap}\ \begin{array}{c} \alpha^x \\ \vdots \\ \beta \\ \hline (\alpha \multimap \beta) \end{array}\ x \qquad\qquad \mathrm{E}_{\multimap}\ \begin{array}{c} \alpha \quad (\alpha \multimap \beta) \\ \hline \beta \end{array}$$

$$\mathrm{I}_{\otimes}\ \begin{array}{c} \alpha \quad \beta \\ \hline (\alpha \otimes \beta) \end{array} \qquad\qquad \mathrm{E}_{\otimes}\ \begin{array}{c} \begin{array}{c} (\alpha \otimes \beta) \\ \hline \alpha \quad\ \beta \end{array} \\ \vdots \\ \hline \gamma \end{array}$$

$$\mathrm{I}_{1\parr}\ \begin{array}{c} (\alpha^\perp)^x \\ \vdots \\ \beta \\ \hline (\alpha \parr \beta) \end{array}\ x \qquad\qquad \mathrm{E}_{1\parr}\ \begin{array}{c} (\alpha^\perp) \quad (\alpha \parr \beta) \\ \hline \beta \end{array}$$

$$\mathrm{I}_{2\parr}\ \begin{array}{c} (\beta^\perp)^x \\ \vdots \\ \alpha \\ \hline (\alpha \parr \beta) \end{array}\ x \qquad\qquad \mathrm{E}_{2\parr}\ \begin{array}{c} (\beta^\perp) \quad (\alpha \parr \beta) \\ \hline \alpha \end{array}$$

$$\mathrm{I}_{!}\ \begin{array}{c} \alpha \\ \hline (!\alpha) \end{array} \qquad\qquad \mathrm{E}_{1!}\ \begin{array}{c} (!\alpha) \\ \hline \alpha \end{array}$$

$$\text{Duplication}_! \quad \dfrac{\begin{array}{c}\dfrac{(!\alpha)}{(!\alpha) \qquad (!\alpha)}\\ \vdots\end{array}}{\beta} \qquad\qquad \text{E}_{2!} \quad \dfrac{(!\alpha) \quad \beta}{\beta}$$

$$\text{I}_? \quad \dfrac{\alpha}{(?\alpha)} \qquad\qquad \text{E}_? \quad \dfrac{(?\alpha) \quad \begin{array}{c}\alpha^x\\ \vdots\\ \beta\end{array}}{\beta}\, x$$

$$\text{I}_\perp \quad \dfrac{\begin{array}{c}\alpha^x\\ \vdots\\ \perp\end{array}}{(\alpha^\perp)}\, x \qquad\qquad \text{E}_\perp \quad \dfrac{\begin{array}{c}(\alpha^\perp)^x\\ \vdots\\ \perp\end{array}}{\alpha}\, x$$

$$\text{Contradiction} \quad \dfrac{\alpha \quad (\alpha^\perp)}{\perp}$$

Hypotheses are as assumptions but they are discharged by inference rules. A hypothesis appears in a rule with an index. This index is also indicated at the right side of the respective rule marking the point where the hypothesis is discharged. Hypotheses neither can be conclusions of inference rules nor are part of the set of (always undischarged) assumptions.

Deductions in NDMF can be described as a graph where vertices are wffs and edges are applications of inference rules on wffs.

We can define the concept of path as a sequence of wffs $\alpha_1, \alpha_2, \ldots, \alpha_n$ in a deduction such that, for a wff α_i, $i > 1$, of this path α_{i-1} occurs immediately above α_i in the deduction, that is, α_{i-1} is one of the premises of the rule that derives α_i.

The concept of essentially modal formulas was used by Prawitz in [Pra1965] to define natural deduction rules for modal operators in S4 and S5 logics. We will adapt this concept to define the exponential rules for "!" and "?".

The inductive definition of the essentially !-modal formula is the following:

i) $(!\alpha)$ is an essentially !-modal formula.
ii) $((?\alpha)^\perp)$ is an essentially !-modal formula.
iii) If α and β are essentially !-modal formulas, so are $(\alpha \otimes \beta)$.
iv) None of the wffs, other than those obtained by i), ii), and iii), can be considered essentially !-modal formulas.

The inductive definition of the essentially ?-modal formula is the following:

i) $(?\alpha)$ is an essentially ?-modal formula.
ii) $((!\alpha)^\perp)$ is an essentially ?-modal formula.
iii) If α and β are essentially ?-modals formulas, so are $(\alpha \parr \beta)$.
iv) None of the wffs, other than those obtained by i), ii), and iii), can be considered essentially ?-modal formulas.

The restrictions to the NDMF inference rules are presented as follows:

i) The rule $\mathrm{I}_{\multimap}$ discharges one, and only one, occurrence of the hypothesis α^x.

ii) In the rule $\mathrm{E}_{\otimes}$, let Σ be the deduction that begins with α and β and ends with γ.

$$\Sigma = \quad \dfrac{\alpha \quad \vdots \quad \beta}{\gamma}$$

In Σ, none of the rules can discharge hypotheses on which $(\alpha \otimes \beta)$ depends. Let r be the last rule of Σ. r infers γ and it is the first rule that links the branches which begins with α and β. Therefore, r must be one of the following rules: "$\mathrm{E}_{\multimap}$", "$\mathrm{I}_{\otimes}$", "$\mathrm{E}_{1\parr}$", "$\mathrm{E}_{2\parr}$", "$\mathrm{E}_{2!}$" or "Contradiction".

iii) The rule $\mathrm{I}_{1\parr}$ discharges one, and only one, occurrence of the hypothesis $(\alpha^{\perp})^x$.

iv) The rule $\mathrm{I}_{2\parr}$ discharges one, and only one, occurrence of the hypothesis $(\beta^{\perp})^x$.

v) In the rule $\mathrm{I}_{!}$, let Σ be the deduction that ends with α.

$$\Sigma = \quad \begin{array}{c}\vdots \\ \alpha\end{array}$$

Let C be any path between α and a hypothesis in Σ on which α depends. Then, the rule $\mathrm{I}_{!}$ can only be applied to the premise α if, for every C, there exists a wff β belonging to C such that β is essentially !-modal.

vi) In the rule Duplication$_{!}$, let Σ be the deduction that begins with two occurrences of $(!\alpha)$ and ends with β.

$$\Sigma = \quad \dfrac{(!\alpha) \quad \vdots \quad (!\alpha)}{\beta}$$

In Σ, none of the rules can discharge hypotheses on which the premise $(!\alpha)$ depends. Let r be the last rule of Σ. r infers β and it is the first rule that links the branches which begins with two occurrences of $(!\alpha)$. Therefore, r must be one of the following rules: "$\mathrm{E}_{\multimap}$", "$\mathrm{I}_{\otimes}$", "$\mathrm{E}_{1\parr}$", "$\mathrm{E}_{2\parr}$", "$\mathrm{E}_{2!}$" or "Contradiction".

vii) In the rule $\mathrm{E}_{?}$, let Σ be the deduction that begins with α^x and ends with β.

$$\Sigma = \quad \begin{array}{c}\alpha^x \\ \vdots \\ \beta\end{array}$$

Let C be any path between β and a hypothesis in Σ on which β depends but α^x. C should contain an essentially !-modal wff γ. Besides of that, β should be essentially ?-modal or β is $\perp$.

viii) The rule $\mathrm{I}_\perp$ discharges one, and only one, occurrence of the hypothesis α^x.
ix) The rule $\mathrm{E}_\perp$ discharges one, and only one, occurrence of the hypothesis $(\alpha^\perp)^x$.

We can observe that $(\alpha^\perp)$ can be represented by $(\alpha \multimap \perp)$. In this case, we can replace "$\mathrm{E}_\multimap$" by the "Contradiction" rule.

In the calculus proposed in [Med2001], there are no rules for the connectives "$\parr$" and "$^\perp$", the operator "?" and the constant "1". They are included by definition. In her work, rules for "$\mathrm{E}_\otimes$" and "Duplication$_!$" follow the pattern of "$\mathrm{E}_\vee$" classical natural deduction rule [Gen1934]. Moreover, the rules for operator "!" do not follow the style proposed by Prawitz [Pra1965] for the necessity modal operator in S4.

3.2 Meaning of the connectives

Using NDMF, we intend to have a better understanding of the linear connectives. In our system, we must discharge an occurrence of an hypothesis once, and only once. This indicates that an hypothesis is treated as a limited and relevant resource.

The connective "$\parr$" is one of the hardest to understand. With the rules "$\mathrm{I}_{1\parr}$", "Contradiction" and "$\mathrm{E}_\perp$", we can prove the axiom $((\alpha \multimap \beta) \multimap ((\alpha^\perp)\parr\beta))$. This axiom clarifies the meaning of "$\parr$". It says that an action of the type $(\alpha \multimap \beta)$, if performed, can generate an action of the type $((\alpha^\perp)\parr\beta)$ which means that we do not have α anymore and we certainly have β. Thus, although "$\parr$" is a disjunction, "$\parr$" carries a conjunctive idea.

We have defined two multiple-conclusion rules: "$\mathrm{E}_\otimes$" and "Duplication$_!$". However, proofs, in our system, have only one conclusion. Thus, every multiple-conclusion rule is part of a cycle in the graph representation of a proof. Our rule "$\mathrm{E}_\otimes$" is similar to the rule "$\mathrm{E}_\wedge$" in the multiple-conclusion calculus for classical logic proposed by Ungar [Ung1992]. Nevertheless, in the calculus proposed by Ungar, the rule "$\mathrm{E}_\wedge$" generates two independent conclusions. On the contrary, in our system, we have to deal with both conclusions of the rule "$\mathrm{E}_\otimes$" in a dependent way reflecting the disjunctive flavor of "$\otimes$".

The rule "Duplication$_!$" has the same structure of the rule "$\mathrm{E}_\otimes$". The formula $(!\alpha)$ represents a (finite) formula $(\ldots((\alpha \otimes \alpha) \otimes \alpha)\ldots \otimes \alpha)$. Thus, with Duplication$_!$, if we have $(!\alpha)$, then we have an unlimited amount of α

Another interesting rule is "$\mathrm{E}_{2!}$". It shows that a resource α, if preceded by "!", may be discharged.

The rules for the operators "!" and "?" of our system were inspired in the rules proposed by Prawitz [Pra1965] for the modal operators "$\Box$" (necessity) and "$\Diamond$" (possibility) of the modal logic S4. The restrictions of the rule "$\mathrm{I}_\Box$" in the Prawitz's system, for instance, were our starting point to draw the restrictions that we have proposed for the rule "$\mathrm{I}_!$". Those restrictions are important to preserve correctness and to allow normalization. Similar observations can be said about the restrictions of the rule "$\mathrm{E}_?$". They were inspired in the rule "$\mathrm{E}_\Diamond$" of the Prawitz's system. Thus, we can say that the operators ! and ? carry a modal idea.

4 Equivalence between NDMF and the Sequent Calculus

The length of a deduction Π in NDMF, denoted by $l(\Pi)$, is defined inductively in the following way:

i) If Π has just one hypothesis, then $l(\Pi) = 1$.
ii) If Π has the form:

$$\frac{\Sigma}{\alpha} \qquad \text{or} \qquad \frac{\Sigma_1 \quad \Sigma_2}{\alpha}$$

then the length of Π is $l(\Pi) = l(\Sigma) + 1$ or $l(\Pi) = max\{l(\Sigma_1), l(\Sigma_2)\} + 1$ [2] respectively.

The length of a deduction Π in sequent calculus, denoted by $l_s(\Pi)$, is defined inductively in the following way:

i) If Π has just the rules of "Identity", "Contradiction" or "Tautology", then $l_s(\Pi) = 1$.
ii) If Π has the form:

$$\frac{\Sigma}{\Gamma \vdash \Delta} \qquad \text{or} \qquad \frac{\Sigma_1 \quad \Sigma_2}{\Gamma \vdash \Delta}$$

then the length of Π is $l_s(\Pi) = l_s(\Sigma)+1$ or $l_s(\Pi) = max\{l_s(\Sigma_1), l_s(\Sigma_2)\}+1$ respectively.

Lemma 1. *If α is an essentially $!$-modal formula, then $\alpha \vdash (!\alpha)$ is proved in the sequent calculus.*

Lemma 1 can be proved by induction on the number of occurrences of $\otimes$ in α.

Lemma 2. *If α is an essentially $?$-modal formula, then $(?\alpha) \vdash \alpha$ is proved in the sequent calculus.*

Lemma 2 can be proved by induction on the number of occurrences of $\parr$ in α.

Theorem 3. (Correctness) *If $\Gamma, (\alpha_1^\perp), \ldots, (\alpha_{n-1}^\perp) \vdash_{NDMF} \alpha_n$, then the sequent $\Gamma \vdash \alpha_1, \ldots, \alpha_n$ is proved in the sequent calculus.*

Theorem 3 can be proved using Lemmas 1 and 2 and by induction on the length of a deduction Π in the system NDMF for $\Gamma, (\alpha_1^\perp), \ldots, (\alpha_{n-1}^\perp) \vdash_{NDMF} \alpha_n$.

Theorem 4. (Completeness) *Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$, eventually empty, and let $\Delta^\perp = \{(\alpha_1^\perp), \ldots, (\alpha_n^\perp)\}$. If $\Gamma \vdash \Delta$ is a sequent proved in sequent calculus, then $\Gamma, \Delta^\perp \vdash_{NDMF} \perp$.*

Theorem 4 can be proved by induction on the length of a deduction Π in the sequent calculus with end sequent as $\Gamma \vdash \Delta$.

Corollary 5. *The NDMF calculus is equivalent to Girard's sequent calculus.*

The proof of Corollary 5 is a consequence of Theorems 3 and 4.

[2] max is a function that returns the greatest value among the input values.

5 Conclusion

In this paper, a natural deduction system was proposed for the multiplicative fragment of linear logic. Our system has introduction and elimination rules for all connectives and operators of this fragment. We proved that our system is equivalent to Girard's multiplicative sequent calculus system.

The rules for the connective "$\parr$" should be improved since they do not follow the Gentzen [Gen1934] idea that rules must deal with just one connective. "$\mathrm{E}_{\parr}$", for example, could follow the pattern of "$\mathrm{E}_{\otimes}$".

$$\mathrm{E'}_{\parr} \qquad \frac{\begin{array}{c}\dfrac{(\alpha \parr \beta)}{\alpha \qquad \beta}\\ \vdots\end{array}}{\gamma}$$

And "$\mathrm{I}_{\parr}$" could follow the pattern of the "$\mathrm{I}_{\otimes}$"

$$\mathrm{I'}_{\parr} \qquad \frac{\alpha \quad \beta}{(\alpha \parr \beta)}$$

In order to preserve the correctness of the system, we must introduce restrictions on the deduction graphs to avoid cycles that begin with a rule "$\mathrm{E'}_{\parr}$" and end with one of the rules: "$\mathrm{E}_{\multimap}$", "$\mathrm{I}_{\otimes}$", "$\mathrm{E}_{2!}$" and "Contradiction". In others words, a cycle that begins with the rule "$\mathrm{E'}_{\parr}$" can end only with the rule "$\mathrm{I'}_{\parr}$". Ungar [Ung1992] defined some restrictions on cycles for his multiple-conclusion classical calculus like those presented above. However, those modified rules for "$\parr$" cannot assure the completeness. It is necessary to add others rules to preserve completeness.

We also intend to extend our calculus for the additive fragment and prove normalization results. Such proofs will allow the implementation of theorem provers for linear logic, which can be used in several artificial intelligence applications where resources are limited.

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